On string backgrounds and (logarithmic) CFT[1]

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Abstract

We discuss the link between string backgrounds and the associated world-sheet CFTs. In the search for new backgrounds and CFTs, Penrose limits and Lie algebra contractions are important tools. The Nappi-Witten construction and the recently discovered logarithmic CFT by Bakas and Sfetsos, are considered as illustrations. We also speculate on possible extensions.

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1 Introduction

A common problem in string theory is to understand the links between the world-sheet description and the space-time (or target-space) physics. A main focus here will be on a particular aspect of this, namely the study of which string backgrounds may be associated to which world-sheet conformal field theories (CFTs). As will be discussed, an intriguing observation is that logarithmic CFT seems to enter the game [1]. We refer to [2, 3] for recent reviews on logarithmic CFT.

Often the link is made explicit by considering the string theory as a (non-linear) σ-model with a given background metric. It is then identified with some sort of Wess-Zumino-Witten (WZW) model on the world sheet. Depending on the WZW model being based on a (non-)semi-simple group, (non-)compact or perhaps a coset in terms of a gauged WZW action, we will get different string backgrounds. Work related to this may be found in [4, 5, 6, 7, 8, 9, 10, 11, 12, 1, 13] and references therein. It turns out that some of the more exotic constructions on the world sheet can be obtained by considering so-called Lie algebra contractions of the Lie algebras underlying some more conventional initial WZW models. Lie algebra contractions are sometimes referred to as Inönü-Wigner or Saletan contractions. One could also consider more general CFTs where the Lie algebra contractions are replaced by linear, but singular, maps of the set of primary fields. The new model (which may not even be an ordinary CFT) is obtained by considering the singular limit. We suggest to refer to these more general constructions as operator product algebra (OPA) contractions.

On the space-time side, we can obtain new geometries by considering Penrose limits of existing geometries. As a result, one obtains pp-wave backgrounds which –in a sense– differ only minimally from flat backgrounds. The reason is that the curvature effects are rendered controllable by the existence of a covariantly constant null Killing vector. We refer to [14] for a recent review on plane waves and their applications.

The aim of this talk is to indicate how Lie algebra or OPA contractions and Penrose limits may be seen as going hand in hand, by presenting a couple of examples based on [4, 1] and some speculations. As a recent offspring of their affair, we recognize the emergence of a logarithmic CFT [1].

2 String backgrounds and CFT

To get an impression of how the link between the world sheet and string background is established, let us consider the ordinary WZW model based on an action like

$$S \;=\; \frac{1}{4\pi}\int_\Sigma (g^{-1}dg)^2 \;+\; \frac{i}{6\pi}\int_B (g^{-1}dg)^3 \tag{1}$$

where B is a three-dimensional space with boundary given by the two-dimensional surface Σ. g takes values in a Lie group, so the one-forms can be expressed in terms of the Lie

algebra generators:

$$g^{-1}\partial_\alpha g \;=\; A^a_\alpha J_a \;, \qquad [J_a, J_b] \;=\; f_{ab}{}^c J_c \tag{2}$$

For the model to be well-defined, we need a bilinear and symmetric two-form, Ω, on the Lie algebra satisfying

$$f_{ab}{}^d\Omega_{cd} + f_{ac}{}^d\Omega_{bd} \;=\; 0 \tag{3}$$

This is referred to as invariance and corresponds to imposing the Jacobi identities on the affine extension of the Lie algebra. Furthermore, the two-form must be non-degenerate, that is, it must be invertible. With all this satisfied, we can rewrite the action (1) as

$$S \;=\; \frac{1}{4\pi}\int_\Sigma d^2x\Omega_{ab}A^a_\alpha A^{b\alpha} \;+\; \frac{i}{12\pi}\int_B d^3x\epsilon^{\alpha\beta\gamma}A^a_\alpha A^b_\beta A^c_\gamma\Omega_{cd}f_{ab}{}^d \tag{4}$$

The two-form Ω trivially exists when G is semi-simple and is then given by the Cartan-Killing form, the trace in the adjoint representation. For non-semi-simple groups, this form is degenerate. Nappi and Witten provided an example in [4] of a WZW model based on a non-semi-simple group admitting a non-degenerate two-form. We shall return to it below.

Now, choosing a parametrization of the group elements as

$$g \;=\; e^{j^aJ_a}...e^{j^bJ_b} \tag{5}$$

one can write the WZW action in terms of the coefficients j^a and their derivatives. This is then compared to the σ-model

$$S_\sigma \;=\; \int\left(G_{MN}\partial_\alpha X^M\partial^\alpha X^N + iB_{MN}\epsilon_{\alpha\beta}\partial^\alpha X^M\partial^\beta X^N\right) \tag{6}$$

where G_{MN} gives the space-time background metric, whereas B_{MN} is the anti-commuting tensor field. Thus, considering the string theory as described by such a σ-model, this provides the link between the world-sheet CFT and the string background. Notice that the dimension of the background is given by the dimension of the Lie group.

Different representations of the group elements (5) will result in different metrics, so the Campbell-Baker-Hausdorff formula is seen to generate coordinate transformations on the background.

An important result of Nappi and Witten was that a sufficiently funky choice of WZW model could lead to a plane-wave background. In [4] they considered a central extension of the two-dimensional Poincare or Euclidean algebra E_2^c:

$$[J, P_i] = \epsilon_{ij}P_j \;, \quad [P_i, P_j] = \epsilon_{ij}T \;, \quad [T, J] = [T, P_i] = 0 \tag{7}$$

Here J generates rotations while P_i generate translations. T is a central element and governs the extension. In the order (P_1, P_2, J, T), the most general two-form reads

$$\Omega \;=\; k\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{8}$$

and is of Lorentz signature $(+++-)$. With the representation

$$g \;=\; e^{x_1 P_1} e^{uJ} e^{x_2 P_1 + vT} \tag{9}$$

the associated metric is worked out to be

$$\frac{1}{k} ds^2 \;=\; dx_1^2 + dx_2^2 + 2\cos(u) dx_1 dx_2 + 2du dv + b du^2 \tag{10}$$

This is recognized as the metric of a plane wave.

One can now address the conformal invariance of the model by showing that the one-loop β-function vanishes, fixing the central charge to 4. One way of checking this non-perturbatively is to consider the generalized Sugawara construction by evaluating the so-called Virasoro master equation. A more general argument for this background to be a good choice in string theory is due to Horowitz and Steif [15], who found that a broad class of pp-waves, being solutions to the supergravity equations, do not receive α' corrections.

Here, instead, we turn to the affair of Penrose limits and Lie algebra contractions. Let us consider the WZW model based on $SU(2) \times \mathbb{R}$ where the second factor is generated by a time-like coordinate, y. In this case one can write the metric as

$$\frac{2}{k'} ds^2 \;=\; d\theta_L^2 + d\theta_R^2 + d\phi^2 + 2\cos(\phi) d\theta_L d\theta_R - dy^2 \tag{11}$$

where θ and ϕ are angles parametrizing $SU(2)$. For $\epsilon \neq 0$, the coordinate transformation

$$k' = 2k/\epsilon \ , \quad \theta_L = \sqrt{\epsilon} x_1 \ , \quad \theta_R = \sqrt{\epsilon} x_2 \ , \quad \phi = \epsilon v + u \ , \quad y = (1 - \epsilon b/2) u \tag{12}$$

is merely a linear transformation with a singularity at $\epsilon = 0$. However, if we consider the correlated limit where

$$\epsilon \longrightarrow 0 \ , \qquad 2k = k'\epsilon \ \text{ fixed} \tag{13}$$

the geometry changes and we end up with the Nappi-Witten (NW) background (10). A correlated limit like (12) and (13) is called a Penrose limit.

This construction has an algebraic analogue on the world sheet. To see this, let us consider the algebra $su(2) \oplus u(1)$ with generators normalized as

$$[J_x, J_y] = J_z \ , \quad [J_y, J_z] = J_x \ , \quad [J_z, J_x] = J_y \ , \quad [U, J_*] = 0 \tag{14}$$

After the following change of basis

$$\begin{pmatrix} P_1 \\ P_2 \\ J \\ T \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1 & \frac{-b}{2a} \\ 0 & 0 & 0 & b \end{pmatrix} \begin{pmatrix} J_x \\ J_y \\ J_z \\ U \end{pmatrix} \tag{15}$$

one can easily write down the commutators of the new generators. Even though the matrix is singular in the limit $a \to 0$, the resulting algebra nevertheless makes sense.

This procedure is an example of a Lie algebra contraction. One finds that the new algebra is E_2^c, the algebra underlying the NW construction.

We have thus illustrated the following schematic relation between string backgrounds and world-sheet CFT, governed by the interpretation of the string theory as a σ-model:

<u>SPACETIME</u>	(σ-model interpretation) $< - - - - - - - - - - - >$	<u>WORLD SHEET</u>
geometry		CFT (WZW model)
$\vert$		$\vert$
$\vert$		$\vert$
correlated		*Lie alg.*
(*Penrose*) *limit*		*contraction*
$\vert$		$\vert$
$\downarrow$		$\downarrow$
geometry’	$< - - - - - - - - - - - >$	CFT’
(NW plane-wave)	(σ-model interpretation)	(non-semi-simple WZW)

The two mediators are correlated (Penrose) limits and Lie algebra contractions, respectively.

3 String backgrounds and logarithmic CFT

Another and more recent example illustrating the general picture above is due to Bakas and Sfetsos [1]. Extensions are discussed in [13, 16]. It is based on a parafermionic model $SU(2)_N/U(1)_N$ times a time-like boson generating $U(1)_{-N}$. Its metric can be written as

$$\frac{1}{N}ds^2 \;=\; -dt^2 + d\theta^2 + \cot^2(\theta)d\phi^2 \tag{16}$$

where t represents the time-like coordinate. The remaining part stems from an Euler-angle representation of an $SU(2)$ WZW model gauged by a $U(1)$ subgroup. The correlated limit of our interest is based on the transformation

$$\theta = \epsilon v + u \;, \qquad t = u \;, \qquad \phi = \sqrt{\epsilon}x \;, \qquad N\epsilon = 1 \tag{17}$$

In the limit $\epsilon \to 0$ the metric becomes

$$ds^2 \;=\; 2dudv + \cot^2(u)dx^2 \tag{18}$$

describing a plane-wave background.

On the world sheet, we start with a level-N parafermionic CFT [17] based on the coset $SU(2)_N/U(1)_N$. It is generated by the two basic fields ψ_1 and $\psi_1^\dagger$. Their conformal weights and the central charge are

$$\Delta(\psi_1^{(\dagger)}) = 1 - \frac{1}{N} \ , \qquad c = \frac{2(N-1)}{N+2} \tag{19}$$

We shall not go into details of the structure of the operator product algebra nor the computation of correlators. In the naive limit with N approaching infinity, the fields become bosons of dimension 1 and the central charge is 2. Combined with an extra $U(1)$ factor as above, we would then have three bosons and $c = 3$.

This parafermionic model has a classical counterpart [18] described by the target-space coordinates appearing in (16). In the classical model the transformation (17) corresponds to a field transformation. Motivated by this observation, Bakas and Sfetsos considered the linear map

$$\Phi = \epsilon \left(\frac{\sqrt{N}}{2}(\psi_1 + \psi_1^\dagger) - J \right), \qquad \Psi = \frac{\sqrt{N}}{2}(\psi_1 + \psi_1^\dagger) + J \ , \qquad P = \sqrt{\epsilon}\frac{\sqrt{N}}{2i}(\psi_1 - \psi_1^\dagger) \tag{20}$$

where J is the $U(1)$ field. All three fields in (20) are seen to be self-conjugate. It is emphasized that Φ and Ψ do not have well-defined weights for finite N – only in the limit $N \to \infty$. This is important as it opens up for the possibility that, even in the limit, these fields may not be primary fields after all. Indeed, in the limit one finds that Ψ is a logarithmic partner to the primary field Φ:

$$\begin{aligned} T(z)\Phi(w) &= \frac{\Phi(w)}{(z-w)^2} + \frac{\partial_w \Phi(w)}{z-w} + \mathcal{O}(1) \\ T(z)\Psi(w) &= \frac{\Psi(w) - \frac{1}{2}\Phi(w)}{(z-w)^2} + \frac{\partial_w \Psi(w)}{z-w} + \mathcal{O}(1) \end{aligned} \tag{21}$$

This is a basic feature of a logarithmic CFT where the Virasoro generator T no longer acts diagonally. A canonical situation is illustrated by the (rank-two) Jordan cell

$$L_0|\psi\rangle = \Delta|\psi\rangle + |\phi\rangle \ , \qquad L_0|\phi\rangle = \Delta|\phi\rangle \tag{22}$$

A CFT with such a construction is known to lead to logarithmic dependencies of correlators, hence the term *logarithmic* CFT.

The resulting logarithmic CFT in [1] has central charge 3 and is a new logarithmic CFT. This, of course, is interesting in itself. It also points in the direction of constructing other new models as we shall indicate in the following. Potentially even more important, their work may turn out to provide an application of logarithmic CFT, something we don't have too many of.

Extensions of the work [1] would naturally be built on cosets like

$$(G_N/H_N) \ \times \ U(1)_{-N} \tag{23}$$

where the $U(1)$ factor gives the time coordinate ensuring a Lorentz signature of the target space. Examples are provided in [13] and are based on ordinary Lie groups. We have recently realized that this seems to extend to supergroups as well. The simplest set-up is based on the graded parafermions:

$$(OSp(1|2)_N/U(1)_N) \ \times \ U(1)_{-N} \tag{24}$$

and is discussed in [16]. The graded parafermionic CFT [19] is generated by the four basic fields $\psi_{1/2}$, $\psi_{1/2}^\dagger$, ψ_1 and $\psi_1^\dagger$, the former two being odd. Their dimensions and the central charge are given by

$$\Delta(\psi_{1/2}^{(\dagger)}) = 1 - \frac{1}{4N} \ , \qquad \Delta(\psi_1^{(\dagger)}) = 1 - \frac{1}{N} \ , \qquad c = \frac{-3}{2N+3} \tag{25}$$

The construction mimics the bosonic scenario above, and using conjugacy as a guiding principle, there is hardly any freedom when generalizing (20). We thus suggest to supplement (20) by the additional linear map

$$\Upsilon^+ = \frac{N^{1/4}}{\sqrt{2}} \left(\psi_{1/2} + \psi_{1/2}^\dagger\right) \ , \qquad \Upsilon^- = \frac{N^{-1/4}}{\sqrt{2}i} \left(\psi_{1/2} - \psi_{1/2}^\dagger\right) \tag{26}$$

The operator product algebra and the structure of correlators in the logarithmic CFT emerging in the limit $N \to \infty$, are discussed in [16]. This logarithmic CFT is generated by five spin-one fields, based on (20) and (26), two of which are odd, and the central charge is 1.

4 Speculations

There are many interesting problems to look at in connection with this subject, and the literature already contains numerous results. Here we list a few speculations with only scarce reference to the existing literature.

• It would be interesting to see how far one could stretch the relationship between Lie algebra or OPA contractions and correlated limits of linear, but singular, coordinate transformations[1]. They are both formally of the form

$$\begin{pmatrix} \text{new} \\ \text{set} \end{pmatrix} = \begin{pmatrix} \text{singular} \\ \text{matrix} \end{pmatrix} \begin{pmatrix} \text{old} \\ \text{set} \end{pmatrix} \tag{27}$$

[1]Before performing this transformation, one may wish to 'prepare' by performing an ordinary and invertible, but not necessarily linear, coordinate transformation. This could be advantageous and would not change the geometry.

mapping the old set of generators (or fields) or coordinates into the new set. Since the number of generators[2] is equal to the dimension of the geometry, one could compare 'directly' the two matrices.

• Another problem is to understand how brane configurations change under the action of taking Penrose limits. Some progress in this direction has been made in [20], for example. In this realm, one could also wonder about the fate of duality under Penrose limits where correlated limits of two dual descriptions would be considered.

• The final point addressed here concerns a possible 'supersymmetric extension' of the NW construction. The basic observation is that the so-called non-reductive $N = 4$ superconformal algebra (SCA) found in [21] contains (an affine extension of) E_2^c as a bosonic subalgebra[3]. This SCA was constructed as a non-trivial Lie algebra contraction of the well-known large $N = 4$ SCA [22], further supporting the idea that it may be of relevance in the present context. It is also linked to the AdS/CFT correspondence as the non-reductive $N = 4$ SCA also contains the so-called asymmetric $N = 4$ SCA [23] as a subalgebra. This asymmetric $N = 4$ SCA was constructed as a superconformal extension of the Virasoro algebra generating the conformal transformations on the boundary of AdS_3.

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[2] In the case of a world-sheet WZW model based on an ordinary Lie group, this number equals the dimension of the Lie group, as already mentioned.

[3] The full structure of the SCA fixes $b = 1$ in the two-form (3) of its subalgebra, and with k playing the role of the level of the affine extension, it is given in terms of the central charge of the SCA: $k = \frac{c_{SCA}}{12}$.

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